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A DUALITY PROPERTY FOR THE SET OF ALL FEASIBLE SOLUTIONS TO AN INTEGER PROGRAM*

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It is shown how to transform the set of all feasible solutions to an integer program represented by a system of linear diophantine inequalities into an 'equivalent' set represented by a system of linear diophantine equations and congruences. A similar transformation is given, working in the opposite direction (i.e. from a system of equations to a system of inequalities).

1. Introduction

Many papers in the literature are concerned with the problem of reducing integer programs from one form to another equivalent form, see e.g. [3, 4, 5, 6]. Such reductions are often usefull as some forms are easier to solve than others. In this paper we will prove (algorithmically) a duality principle showing a correspondence between sets of constraints expressed in the form of linear inequalities. Specifically we will show that under very general conditions the following transformations are possible:

(1) Given a convex set in n-dimensional Euclidean space represented by $m>n$ linear inequalities with integral coefficients.

An I-equivalent convex set (i.e. there exists an explicit 1–1 correspondence between the (integral) points in the first set and the (integral) points in the second) can be constructed in the m-dimensional Euclidean space, represented by a system consisting of $m-n$ diophantine linear equations and up to n linear congruences with integral coefficients.

(2) Given a convex set in the n-dimensional Euclidean space, $n\geq 2$, represented by a system of $m<n$ diophantine linear equations and $m_1\geq 1$ linear inequalities with integral coefficients.

An I-equivalent convex set in the $n-m$ dimensional Euclidean space can be constructed, represented by a system of m_1 linear inequalities with integral coefficients.

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2. First reduction theorem

Given a set of m inequalities in n variables

$$\sum_{j=n}^{1} a_{ij}x_j + b_i \geq 0, \quad 1 \leq i \leq m. \tag{1}$$

We represent the above set in the following matrix form

$$(1, x_n, \ldots, x_1)A \geq 0. \tag{2}$$

Then A is $(n+1) \times m$ (with the b's in its first row).

We shall refer to the convex set K defined below

$$K = \{(x_n, \ldots, x_1) \colon (1, x_n, \ldots, x_1)A \geq 0\} \tag{3}$$

and we shall make the following assumptions:

(a) $m \geq n+1$.

(b) The entries of A are all integers and the rank of A is $n+1$.

The following facts are implies by the above assumptions.

Fact 1. *The matrix A has a right inverse A^{-1} (e.g. $A^{-1} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ the Moore–Penrose generalized inverse) whose entries are rational numbers (by property* (b)) *and is $m \times (n+1)$.*

Fact 2. *The manifold $(1, \alpha_n, \ldots, \alpha_1)A = (y_m, \ldots, y_1)$ where the α's are parameters is equivalent to a system of $(m-n)$ simultaneous equations in the variables $y_m, \ldots, y_1$ which can be found easily by standard Gaussian elimination.*

Denote this set of equations by

$$\sum_{j=m}^{1} c_{ij}y_j = d_i, \quad 1 \leq i \leq m-n \tag{4}$$

or in matrix form

$$yC = d. \tag{5}$$

Based on (5) we define the convex set K_1 below

$$K_1 = \{y \colon yC = d,\ y \geq 0\}. \tag{6}$$

Fact 3. *Consider the mapping*

$$y = (1, x)A. \tag{7}$$

It is easy to see that (7) *maps K 1–1 onto K_1 where the inverse mapping is given by $(1, x) = yA^{-1}$ with A^{-1} a right inverse of A (to be assumed in the sequel).*

Fact 4. *Let $(\gamma_{i1} \cdots \gamma_{im})$ be the i-th column of A^{-1}, and let g_i be the l.c.m. (least common multiple) of the denominators of the entries in the column of A^{-1}, $1 \leq i \leq n+1$. Then all the coefficients in the system of equations below are integers*

$$\sum_{j=m}^{1} (\gamma_{ij} g_i) y_j \equiv 0 \ (\mathrm{mod}\ g_i), \quad 2 \leq i \leq n+1. \tag{8}$$

g_i exists as follows from Fact 1.

Fact 5. *Under the mapping* (7), *x in K has integral components if and only if its corresponding y in K_1 satisfies the system* (8) *of congruences and has integral components.*

Proof. If x has integral components, then by (7), y has integral components and, as $(1, x) = yA^{-1}$, we have that

$$\sum \gamma_{ij} y_j = \frac{1}{g_i} \sum \gamma_{ij} g_i y_j = x_i$$

which implies that y satisfies (8). The converse follows from the fact that $(1, x) = yA^{-1}$.

Define the convex set K_2 below

$$K_2 = \{y: yC = d,\ y \geq 0 \text{ and } y \text{ satisfies (8)}\}.$$

Facts 1 to 5 imply the following.

Theorem 1. *The mapping* (7) *is a one-to-one mapping from K to K_2 such that x in K has integral coefficients if and only if its corresponding y has integral coefficients.*

Remark 1. The calculation of $(AA^{\mathrm{T}})^{-1}$ involves operations with large numbers and one should choose, for computational convenience, a different right inverse than the Moore–Penrose generalized inverse.

Remark 2. As is easy to see some of the modular equations (8) can be discarded in the definition of K_2, specifically, those equations corresponding to columns of A^{-1} whose entries are integers; such equations are satisfied by any vector with integral components.

3. An example

Given the set of inequalities:

$$x_2 + 110x_1 \geq 5172,$$

$$6x_2 + 663x_1 \leq 31171,$$

$$5x_2+552x_1\geq 25952,$$

$$x_1\geq 0.$$

The corresponding convex set K is defined by

$$(1,x_2x_1)\begin{bmatrix} -5172 & 31171 & -25952 & 0 \\ 1 & -6 & 5 & 0 \\ 110 & -663 & 552 & 1 \end{bmatrix}\geq 0$$

Denote the above 3×4 matrix by A. It is easy to see that A satisfies conditions (a) and (b). Gaussian elimination will result in the equations (4) below:

$$6y_4+y_3+3y_1=139,$$

$$5y_4-y_2+2y_1=92.$$

We compute now the matrix A^{-1}. One possible solution is:

$$A^{-1}=\begin{bmatrix} 1/47 & 251 & 0 \\ 1/47 & 20 & 0 \\ 1/47 & -26 & 0 \\ 1/47 & 2 & 1 \end{bmatrix}$$

The last two columns of A^{-1} have integral entries so that the modular equations (8) are superfluous for this example. The resulting convex set K_2 is therefore defined by the following constraints

$$6y_4+y_3+3y_1=139,$$

$$5y_4-y_2+2y_1=92,$$

$$y_i\geq 0,\quad 1\leq i\leq 4.$$

Given any vector $(\hat{x}_2,\hat{x}_1)$ of integers in K the vector $(1,\hat{x}_2,\hat{x}_1)A=(\hat{y}_4,\hat{y}_3,\hat{y}_2,\hat{y}_1)$ is a vector of integers in K_1. Given any vector of integers $(\hat{y}_4\hat{y}_3\hat{y}_2\hat{y}_1)$ in K_1, the vector $(\hat{x}_2,\hat{x}_1)$ defined by $(\hat{y}_4\hat{y}_3\hat{y}_2\hat{y}_1)A^{-1}=(1,\hat{x}_2,\hat{x}_1)$ is a vector of integers in K. Thus, e.g. the vector $(20,10,14,3)$ is in the set K_2. The corresponding vector $(20,10,14,3)A^{-1}=(1,4862,3)$ reduces to the vector $(4862,3)$ which is in the set K.

Given two convex sets K_1 and K_2, K_1 will be called I-equivalent to K_2 if there exists a 1–1 mapping from K_1 onto K_2 such that points with integral coordinates in the first set are mapped to points with integral coordinates in the second set, and viceversa.

In the previous section we have shown how to reduce a convex set defined by a system of inequalities to an I-equivalent convex set defined by a system of equations over an expanded space. In the next sections we consider the dual problem, i.e. given a convex set defined by a system of equations, find an I-equivalent convex set defined by a system of inequalities over a reduced space.

4. A useful procedure

The following problem is classical. Given a vector $a=(a_1 \cdots a_n)$, $n \geq 2$, of integers such that $\gcd(a_1, \ldots, a_n)=1$ find a nonsingular matrix A with integral entries such that

$$Aa^{\mathrm{T}}=(1,0,\ldots,0)^{\mathrm{T}}, \quad |A|=1 \tag{9}$$

where $|A|$ denotes the determinant of A.

There are many algorithms for solving the above problem, all of them based on the extended Euclidean algorithm, see e.g. [1,2].

If A is a matrix satisfying (9) above, then A^{-1} has the following properties:

(i) A^{-1} has integral entries (follows from $|A|=1$ and A has integral entries).

(ii) The first column of A^{-1} is equal to a^{T} (a^{T} is the unique solution of the equation $Ax=(1,0,\ldots,0)^{\mathrm{T}}$ which is satisfied by the first column of A^{-1}).

As was shown in [2] the number of arithmetical operations needed for calculating A is $\mathrm{O}(n^2+n\log(\max a_i))$.

For further reference we shall call the algorithm described in [2] Algorithm B.

5. Second reduction theorem

We are able now to prove the following

Theorem. *Given a convex set K represented by a system as defined below, over the n-dimensional Euclidean space, $n \geq 2$.*

$$\sum_{j=n}^{1} u_{ij}x_j+b_i \geq 0, \quad 1 \leq i \leq m, \tag{10.1}$$

$$\sum_{j=n}^{1} u_{ij}x_j+b_i=0, \quad m<i\leq m_1 \text{ with } m<m_1<m+n. \tag{10.2}$$

An I-equivalent convex set K_1 can be found such that K_1 is represented by a system having the following form

$$\sum_{j=n-1}^{1} c_{ij}y_j+d_i \geq 0, \quad 1 \leq i \leq m, \tag{11.1}$$

$$\sum_{j=n-1}^{1} c_{ij}y_j+d_i=0, \quad m<i\leq m_1-1 \tag{11.2}$$

where all the u's, b's, c's and d's are integers and the set (11.2) *of equations in the representation of K_1 is void if $m_1=m+1$.*

Proof. We show first how to construct K_1 and then prove that K_1 is I-equivalent to K.

(1) Delete the last equation from (10.2) and represent the set (10.1) and the remaining set of equations in (10.2) in the form

$$(1,x_n,\dots,x_1)U\geq 0, \quad U \text{ is } (n+1)\times m, \tag{12.1}$$

$$(1,x_n,\dots,x_1)V=0, \quad V \text{ is } (n+1)\times(m_1-m-1). \tag{12.2}$$

(2) Apply Algorithm B described above to the coefficients of the deleted equation (we may assume that the gcd of those coefficients is 1). Denote those coefficients by

$$v_n, v_{n-1},\dots,v_1. \tag{13}$$

The resulting matrix A is $n\times n$ and its inverse A^{-1} has its first column equal to the vector in (13).

(3) Multiply the first row of A by the value $-b_{m_1}$ to get a new matrix $\hat{A}$.

(4) Set

$$U_1=\begin{bmatrix} 1 & \\ 0 & \\ \vdots & \hat{A} \\ 0 & \end{bmatrix} U, \qquad V_1=\begin{bmatrix} 1 & \\ 0 & \\ \vdots & \hat{A} \\ 0 & \end{bmatrix} V. \tag{14}$$

If a column of V_1 is an all zero column, delete that column.

U_1 is $n\times m$ and V_1 is $n\times(m_1-t)$, $t\geq m+1$ and both matrices have integral entries.

(5) The required I-equivalent convex set K_1 is now defined by

$$(1,y_{n-1},\dots,y_1)U_1\geq 0, \tag{15.1}$$

$$(1,y_{n-1},\dots,y_1)V_1=0, \tag{15.2}$$

and the transformation between the two sets is defined by:

given $(x_n,\dots,x_1)$, define $(y_{n-1},\dots,y_1)$ by

$$(x_n,\dots,x_1)A^{-1}=(-b_{m_1},y_{n-1},\dots,y_1).$$

given $(y_{n-1},\dots,y_1)$, define $(x_n,\dots,x_1)$ by

$$(x_n,\dots,x_1)=(1,y_{n-1},\dots,y_1)\hat{A}.$$

To show that K is I-equivalent to K_1 assume first that $(x_n\cdots x_1)=x\in K$, with the x_i's integers. Then

$$xA=\left(\sum_{j=1}^{n} u_{m_1j}x_j, y_{n-1},\dots,y_1\right)=(-b_{m_1},y_{n-1},\dots,y_1)=(-b_{m_1},y),$$

where the y_j's are integers. This follows from the construction of A and from the fact that x satisfies the last equality of (10.2) ($x\in K$). Therefore, $x=(-b_m,y)A=(1y)\hat{A}$, by the definition of $\hat{A}$.

$x\in K$ also implies that

$$(1,x)U \geq 0 \quad \text{or} \quad (1,(1,y)\hat{A})U \geq 0,$$

$$(1,x)V = 0 \quad \text{or} \quad (1,(1,y)\hat{A})V = 0$$

which implies that

$$(1,y)\begin{bmatrix} 1 & \\ 0 & \\ \vdots & \hat{A} \\ 0 & \end{bmatrix} U = (1,y)U_1 \geq 0$$

and

$$(1,y)\begin{bmatrix} 1 & \\ 0 & \\ \vdots & \hat{A} \\ 0 & \end{bmatrix} V = (1,y)V_1 \geq 0$$

showing that $y \in K_1$.

Assume now that $(y_{n-1} \cdots y_1) = y \in K_1$. Reversing the above set of implications we then get that

$$(1,(1,y)\hat{A})U \geq 0 \quad \text{and} \quad (1,(1,y)\hat{A})V = 0$$

or

$$(1,(-b_{m_1},y)U \geq 0 \quad \text{and} \quad (1,(-b_{m_1},y)A)V) = 0.$$

Setting $x = (-b_{m_1}, y)A = (1,y)\hat{A}$ we get that

$$(1,x)U \geq 0 \quad \text{and} \quad (1,x)V = 0. \tag{16}$$

We have also $xA^{-1} = (-b_{m_1}, y)$. From the construction of A^{-1} we know that $xA^{-1} = (\Sigma v_{m_1 j} x_j, y)$. Thus $\Sigma v_{m_1 j} x_j = -b_{m_1}$ which together with (16) implies that $x \in K$. If y is a vector of integers, then by the definition of x and by the construction of A^{-1}, x is a vector of integers. The proof is now complete.

Corollary. *Given a convex set K represented by a system as defined below with integral coefficients*

$$\sum_{j=n}^{1} u_{ij}x_j + b_i = 0, \quad 1 \leq i \leq m, \ n > m, \tag{17.1}$$

$$x_j \geq 0, \qquad 1 \leq j \leq n. \tag{17.2}$$

An I-equivalent convex set K_1 can be found such that K_1 is represented by a system having the form

$$\sum_{j=n-t}^{1} c_{ij}y_j + d_i \geq 0, \quad 1 \leq i \leq n, \ 1 \leq t \leq m \tag{18}$$

with integral coefficients.

Proof. Rewrite (17) to comply with the representation of K in (10):

$$\sum_{j=n}^{1} u_{ij}x_j + b_i \geq 0 \quad \text{with } b_i = 0$$

and

$$u_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases} \quad \text{for } 1 \leq i \leq n. \tag{19.1}$$

Here (19.1) represents (17.2) (the order is changed to comply with the representation given in (10)).

Similarly, we represent (17.1) by

$$\sum_{j=n}^{1} v_{ij}x_j + b_i = 0, \quad n+1 \leq i \leq n+m. \tag{19.2}$$

Apply now the construction given in the proof of the theorem to the above system successively until all the equations (19.2) are removed.

An Example. Given the convex set defined by the following equations

$$\begin{aligned} &5x_4 + x_3 + x_2 - 7 = 0, \\ &2x_4 + 5x_3 + 3x_2 + x_1 - 21 = 0, \\ &3x_4 + 2x_3 + x_2 + x_1 - 18 = 0, \\ &x_i \geq 0. \end{aligned} \tag{20}$$

First reduction. The matrix $\hat{A}_1$ with regard to the third equation is found to be

$$\hat{A}_1 = \begin{bmatrix} 18 & -18 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

The set (20) is therefore I-equivalent to the set

$$[1, y_3, y_2, y_1] \begin{bmatrix} 1 & \\ 0 & \\ 0 & \hat{A}_1 \\ 0 & \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (1, y_3, y_2, y_1)\hat{A}_1 \geq 0 \tag{21.1}$$

together with

$$[1, y_3, y_2, y_1] \begin{bmatrix} 1 & \\ 0 & \\ 0 & \hat{A}_1 \\ 0 & \end{bmatrix} \begin{bmatrix} -7 & -21 \\ 5 & 2 \\ 1 & 5 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} = (1, y_3, y_2, y_1) \begin{bmatrix} 65 & -75 \\ -7 & 11 \\ -3 & 6 \\ -4 & 4 \end{bmatrix} = 0. \tag{21.2}$$

Second reduction. The matrix $\hat{A}_2$ with regard to the second equation in (21.2) is found to be

$$\hat{A}_2=\begin{bmatrix} -75 & 150 & 0 \\ -6 & 11 & 0 \\ 4 & -8 & 1 \end{bmatrix}.$$

The set (21) is therefore I-equivalent to the set

$$(1,z_2,z_1)\begin{bmatrix} 1 & \\ 0 & \hat{A}_2 \\ 0 & \end{bmatrix}\hat{A}_1=(1,z_2,z_1)\begin{bmatrix} 18 & -93 & 150 & 0 \\ 1 & -7 & 11 & 0 \\ -1 & 5 & 8 & 1 \end{bmatrix}\geq 0 \qquad (22.1)$$

together with

$$(1,z_2,z_1)\begin{bmatrix} 1 & \\ 0 & \hat{A}_2 \\ 0 & \end{bmatrix}\begin{bmatrix} 65 \\ -7 \\ -3 \\ -4 \end{bmatrix}=(1,z_2,z_1)\begin{pmatrix} 140 \\ 9 \\ -8 \end{pmatrix}=0. \qquad (22.2)$$

Third reduction. The matrix $\hat{A}_3$ with regard to the remaining equation is found to be

$$\hat{A}_3=\begin{bmatrix} -140 & -140 \\ 8 & 9 \end{bmatrix}.$$

The set (22) is therefore I-equivalent to the set

$$(1,w)\begin{bmatrix} 1 & \\ 0 & \hat{A}_3 \end{bmatrix}\begin{bmatrix} 18 & -93 & 1500 \\ 1 & -7 & 110 \\ -1 & 5 & 81 \end{bmatrix}$$

$$=(1,w)\begin{bmatrix} 18 & 187 & -270 & -140 \\ -1 & -11 & 16 & 9 \end{bmatrix}\geq 0. \qquad (23)$$

For any solution $\hat{w}$ satisfying (23) the corresponding solution to the original set of equations is found by multiplying $(1,\hat{w})$ by the matrix

$$\begin{bmatrix} 18 & 187 & -270 & -140 \\ -1 & -11 & 16 & 9 \end{bmatrix}$$

computed above.

One finds easily that the only solution satisfying (23) is $w=17$ so that the original set (20) has the corresponding solution

$$x_4=18-17=1, \quad x_3=187-17*11=0,$$

$$x_2=-270+17*11=2, \quad x_1=-140+17*9=13,$$

and this is the only solution of (20).

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